General Formulation

Finite elements: Dr Colin Cotter

General formulation

- Fluid dynamics application normally start from differential equations
- Structural mechanics also use point of view of energy of force balance at equilibrium

General formulation

Two approaches to formulating problems:

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Two approaches to formulating problems:

- Method of weighted residuals

General formulation

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- Method of weighted residuals
- Principle of virtual work

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- The choice of the **conditions** which are to be satisfied defines the type of numerical method.
- The method of weighted residuals illustrates how the choice of different weight (or test) functions in an integral or **weak form** of the equation can be used to construct many of the common numerical methods

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When we have the exact answer which satisfies then *R(u)*= 0. This is the only way of ensuring *R(u)* is zero everywhere.

Method of weighted residuals

For a given numerical approximation we don't know the exact form of the residual and so we want to eliminate this term.

We multiply the equation by a weight (test) function, and integrated over the solution region, to obtain

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This is the integral (weak) form of the equation, if true for all *w*

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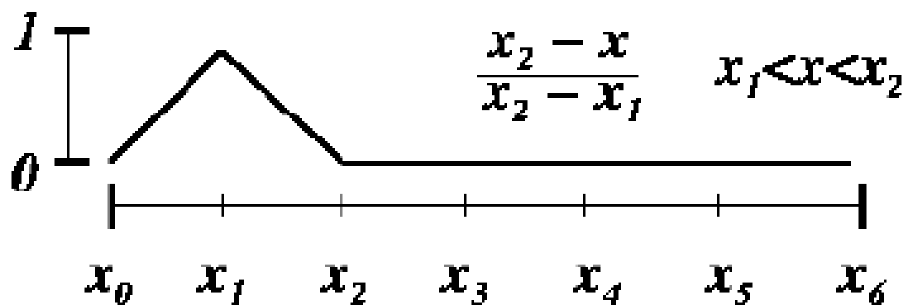

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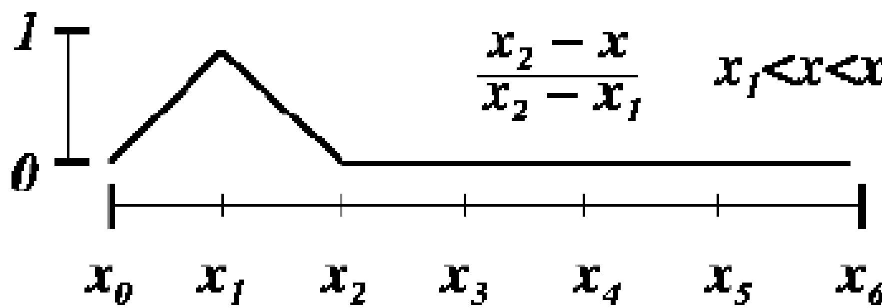

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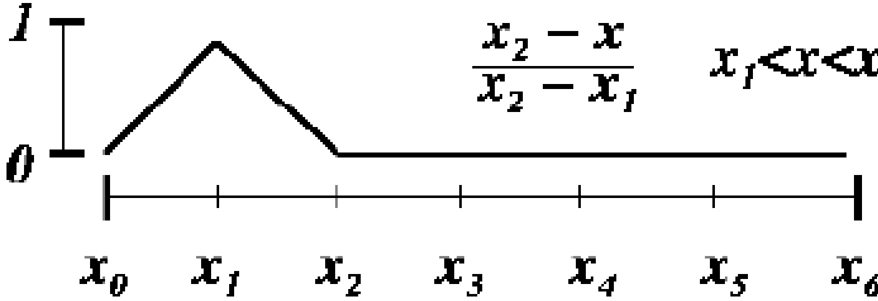

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If we use a different choice of a continuous function so then the projection is referred to as the Petrov-Galerkin methods. This arises when we want to introduce upwinding into the finite element method

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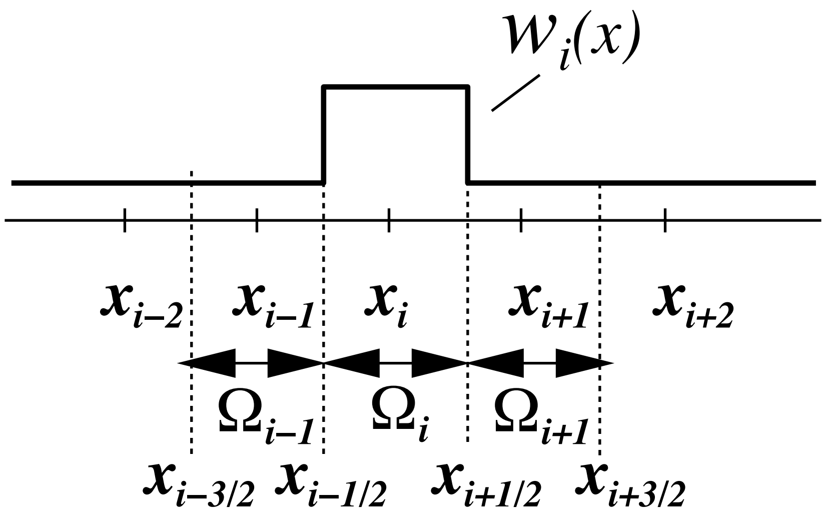

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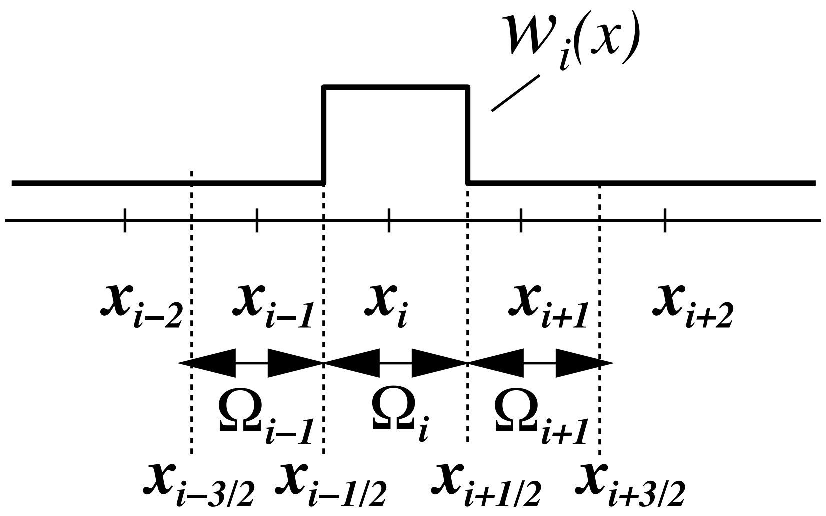

If we choose $w_i(x) = \delta(x - x_j)$ where x_j are the mesh points then we have the collocation method which is the starting point of the finite difference method.

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$$(b^2 - 4ac) > 0 \quad \Rightarrow \quad \textbf{Hyberbolic}$$

$$(b^2 - 4ac) = 0 \quad \Rightarrow \quad \textbf{Parabolic}$$

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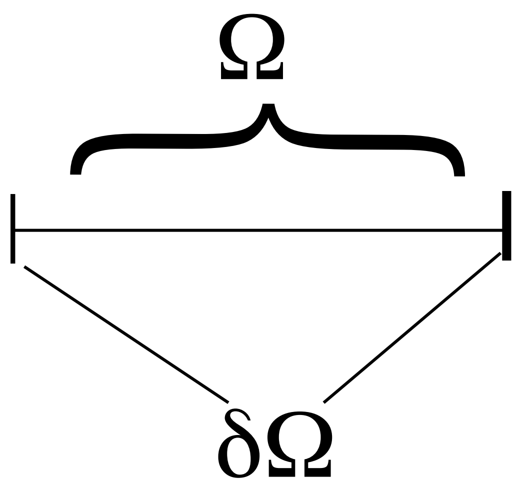
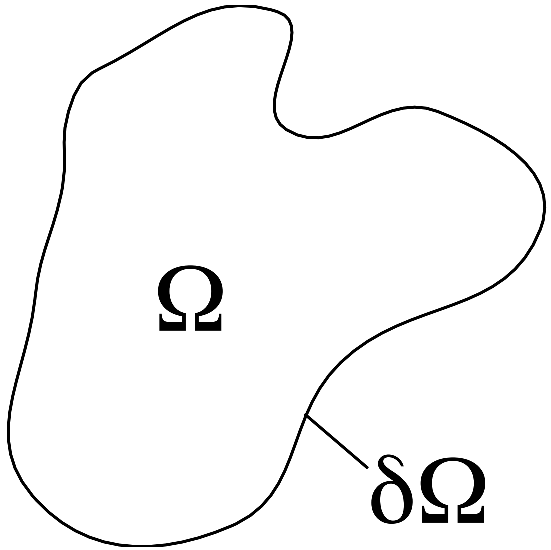
Boundary conditions

For a differential equation to be well posed we need to have appropriate boundary conditions and the same is true for the matrix problem.





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Robin/Mixed	$u(\partial\Omega) + \frac{\partial u}{\partial x}(\partial\Omega) = e$	$u(\partial\Omega) + \frac{\partial u}{\partial n}(\partial\Omega) = h(\partial\Omega)$

Boundary properties

- Different conditions can (and sometimes must) be attached to different parts of the boundary depending on the mathematical properties of the equation.
- For example, if the equation is hyperbolic we must only specify conditions on an inflow boundary.

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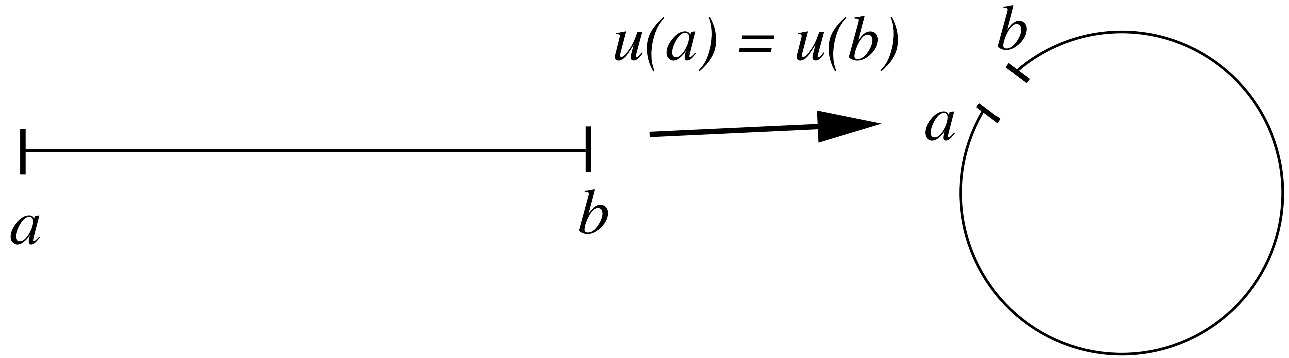

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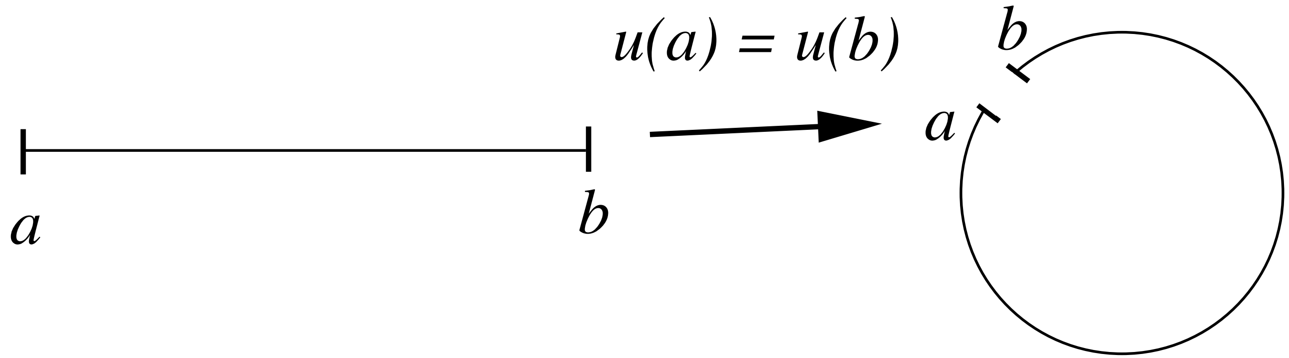

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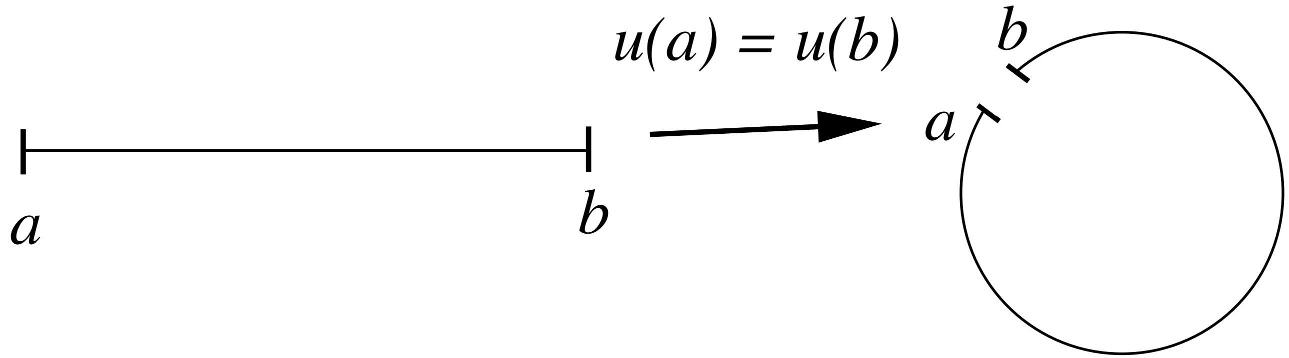

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- A stage of a compressor which is not close to inlet or outlet might also be considered as having periodic boundary conditions.